

# Curious interior solutions of general relativity.

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## Abstract

In this article, we provide a discussion on a composite class of exact static spherically symmetric vacuum solutions of Einstein's equations. We construct the composite solution of Einstein field equation by match the interior vacuum metric in Schwarzschild original gauge, to the exterior vacuum metric in isotropic gauge, at a junction surface. This approach allows us to associate rigorously with both gauges as a same "space", which is a unique differentiable manifold  $M^4$ .

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## 1 Introduction

The field equations in Einstein gravity theory are non linear in nature. For a classical field, the differential equations consist of purely geometric requirements imposed by the idea that space and time can be represented by a Riemannian (Lorentzian) manifold, together with the description of the interaction of matter and gravitation contained in Einstein's equations

$$G_{ab} = T_{ab}, \quad (1)$$

This equation involved a match between a purely geometrical object so called Einstein tensor  $G$ , and an object which depends on the properties of matter the energy-momentum tensor  $T$  which contains quantities like the ordinary density and pressure of matter. Hence, the geometry of 4D spacetime is governed by the matter it contains. However, this split is artificial. According to the standard textbooks the general relativity exhibits general covariance: its laws-and further laws formulated within the general relativistic framework-take on the same form in all coordinate systems [1]. On the other hand Einstein's equations (1) determine the solution of a given physical problem up to four arbitrary functions, i.e., up to a choice of gauge transformations. This theory for definition of concept of co-ordinate system use geometrical terms; meanwhile, the geometrically interpreted co-ordinate system can emerge here only together with the geometry, i.e. with definition of metric tensor  $g_{ab}$  [2]. The variables  $x$ , used in Einstein's equations, represent co-ordinates of points of abstract four-dimensional manifold  $M^4$  over which there are a set pseudo Rimanien spaces  $V^4(g)$ , generated by set of solutions  $g_{ab}$ . Co-ordinates in each of such spaces

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have the specific properties differing from their properties in other spaces [3]. Moreover, a new class of co-ordinates each time is postulated subsystem

$$C(\mu)g_{ab} = 0. \quad (2)$$

where  $C(\mu)$  - some algebraic or differential operators. Specifically four of ten field equations will not be transformed by those or other rules, but simply replaced by hand with the new. In contrast with general relativity, Newtonian theory has as the geometrical foundation the Euclidean space and absolute time. Compared with Newtonian gravity, general relativity has one more independent variable, 9 more dependent ones, the result of these changes being that the general form of the field equations expands to 10 partial differential equations (in terms of the metric and coordinates). Usually used co-ordinate conditions can be wrote in the form of four equations (2). Thereby for any four of components  $g_{ab}$  emerge the relations with remaining six and, probably, any others, known functions. Certainly, equations (2) cannot be covariant for the arbitrary transformations of independent variables, and similarly should not contradict Einstein's equations or to be their consequence.

The choice of a reference frame in a general relativity quite often compare to gauges of potentials in an electrodynamics. But this analogy is the most superficial: this or that gauge is a problem of exclusively convenience, its this or that expedient does not influence in any way on a values of physical quantities and it is not related to observation requirements, - whereas the choice of co-ordinate system is related to all it essentially.

The specification of the energy-momentum tensor played a very important role. The exact solutions known have all been obtained by restricting the algebraic structure of the Riemann tensor, by adding field equations for the matter variables or by imposing initial and boundary conditions. One surprise for the reader may lie in the fact that in a certain gauge any metric whatsoever is a 'solution' of (1) if no restriction is imposed on the energy-momentum tensor, since (1) then becomes just a definition of  $T_{ab}$ . Since the field equations are very complicated, to find solutions physicists makes simplifying assumptions about the left-hand-side or the right-hand-side. Most popular simplifying assumptions about the right-hand-side of (1) are that  $T_{ab}$  represents vacuum. On the other hand, as is well known simplifying assumptions about the left-hand-side often comprise static and spherical symmetry. The most commonly approach employ the assumption that this configurations describes the gravitational field outside any body with spherically symmetric mass distribution. Some of this solutions where discovered at early stage of development of general relativity, but up to now they are often considered as equivalent representation of some "unique" solution. However, the physical and the geometrical meaning of the radial coordinate  $r$  are not defined by symmetry reasons and are unknown a priori [4],[5]. The review by Fiziev outlines that various vacuum spherical solution in different gauges leads to existence infinitely many different static solutions of Einstein equations (1) with spherical symmetry, a point singularity, placed at the center of symmetry, and vacuum outside this singularity, and with the same Keplerian mass  $M$ . This paper will present a new feature of well known solution to the spherically symmetric time independent Einstein system of equations that

govern the behavior of the space-time in the "interior" vacuum Schwarzschild solution. At the outer boundary the solutions will be matched to the external vacuum solution, in a different gauge, for the field equations i.e. the solution in isotropic coordinates.

## 2 Composite solution.

The metric satisfies to those or other co-ordinate conditions if some of quantities  $g_{ab}$  are linked by some relations, - whether it be in any point, on a surface or in four-dimensional domain  $\Omega \subset V^4(g)$ . By definition all co-ordinate systems in manifold  $M^4$  at least locally are equivalent; on the other hand if in  $M^4$  the metric is introduced, properties of functions  $g_{ab}$  in different co-ordinates become different. Let us consider two distinct manifolds  $M^{4+}$  and  $M^{4-}$ . The metric in these manifolds generated by set of solutions of field equations (2) given by  $g_{ab}^+(x^a_+)$  and  $g_{ab}^-(x^a_-)$ , in terms of independently defined coordinate systems  $x^a_+$  and  $x^a_-$ . The manifolds glued at the boundary hypersurfaces  $\Sigma_+$  and  $\Sigma_-$  using independently defining co-ordinates systems  $x^a_{\pm}$ . A common manifold  $M^4$  is obtained by assuming the continuity of four-dimensional coordinates  $x^a_{\pm}$  across  $\Sigma$ , then  $g_{ab}^+ = g_{ab}^-$  is required, which together with the continuous derivatives of the metric components  $\partial g_{ab} / \partial x^c |_+ = \partial g_{ab} / \partial x^c |_-$ , provide the Lichnerowicz conditions [6].

The resulting manifold  $M$  is geodesically complete and possesses two regions connected by a hyper-surface  $\Sigma$ . Since the interior vacuum solution is to be matched with an exterior Schwarzschild solution at the junction surface  $r = a$  we use the Darmois-Israel formalism [7]. Using the field equations, the surface stress-energy tensor can be calculated in terms of the jump in the second fundamental form across  $\Sigma$ . Because  $M$  is piecewise vacuum solution, the Einstein tensor is zero everywhere the stress-energy tensor localized at the junction surface can be calculated

$$T_{ab} = -\delta(\eta)([K_b^a] - \delta_b^a[K]). \quad (3)$$

The extrinsic curvature, or the second fundamental form, is defined as

$$K_{b\pm}^a = \frac{1}{2}g^{ab}\frac{\partial g_{ba}}{\partial \eta}|_{\eta=\pm 0}$$

where  $\eta$  the proper distance away from the  $\Sigma$  and  $[K]$  denotes the trace of  $[K_{ab}] = K_{ab}^+ - K_{ab}^-$ .

The approach to be taken here to the static spherically symmetric relativistic configurations involved a match between solutions of Einstein equations in Schwarzschild original coordinates [8], with solutions in isotropic gauge [9]. The fields equations in this case simply state the metric field is just a field in a spherically symmetric space time. The solution will be given in terms of explicit closed-form functions of the radial coordinate for the three metric coefficients. According to the widespread common opinion, the most common form of line element of a spherically symmetric spacetime in comoving coordinates can be written as

$$ds^2 = -g_{tt}(r, t)dt^2 + g_{rr}(r, t)dr^2 + 2g_{rt}(r, t)drdt + \rho(r, t)^2(d\theta^2 + \sin^2(\theta)d\varphi^2). \quad (4)$$

We are free to reset our clocks by defining a new time coordinate

$$t = t' + f(r).$$

with  $f(r)$  an arbitrary function of  $r$ . This allows us to eliminate the off-diagonal element  $g_{rt}$ . Therefore we shall consider the matching of two static and spherically symmetric spacetimes given by the following line elements

$$ds_{\pm}^2 = -g_{tt}(r, t)_{\pm} dt^2 + g_{rr}(r, t)_{\pm} dr^2 + \rho(r, t)_{\pm}^2 (d\theta^2 + \sin^2(\theta) d\varphi^2). \quad (5)$$

of  $M_{\pm}^4$ , respectively, where  $g_{tt}(r, t)$ ,  $g_{rr}(r, t)$  and  $\rho(r, t)$  are of class  $C^2$ .

In his pioneering article Schwarzschild has used a radial variable  $r$  and the gauge

$$\det \|g_{ab}\| = 1, \quad (6)$$

for the spherically symmetric static metric. He had fixed the three unknown functions

$$g_{tt}(r) = 1 - \frac{2M}{\rho(r)} > 0, g_{rr}(r) = -\frac{1}{g_{tt}} < 0 \quad (7)$$

obtaining

$$\rho(r) = \sqrt[2]{r^3 + \rho_G^3}, \quad (8)$$

Another widespread form of the Schwarzschild's solution, was reached using isotropic gauge (for example in [10]),

$$ds^2 = -\hat{g}_{tt}(r) dt^2 + \hat{g}_{rr}(r) (dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2), \quad (9)$$

Obviously, these co-ordinates may be used as the same co-ordinate for the metric field with various properties in different domain. To begin with, we consider the solutions of field equations for line elements (9) and (6), (7). The result is

$$\begin{aligned} \hat{g}_{tt} &= \frac{\hat{\alpha}(1 - 4r\hat{\beta})^4}{r^4}, \\ \hat{g}_{rr} &= \frac{\hat{\gamma}(1 + 4r\hat{\beta})^2}{(1 - 4r\hat{\beta})^2}, \end{aligned} \quad (10)$$

and

$$\begin{aligned} g_{tt} &= \frac{\alpha r^4}{(r^3 + \rho_G^3)[\alpha(r^3 + \rho_G^3)^{1/3} - \beta]}, \\ g_{rr} &= \alpha - \frac{\beta}{(r^3 + \rho_G^3)^{1/3}}, \end{aligned} \quad (11)$$

where  $\alpha, \beta, \hat{\alpha}, \hat{\beta}, \hat{\gamma}$  arbitrary constants.

Analogous as in stellar models this implies that the one of the solutions we can assume as "interior" but another as "exterior" vacuum solution. Now in order for these line elements to be continuous across the junction we impose an explicit definition for the arbitrary constants in solutions (10), (11). The brief computation yields

$$\begin{aligned}
\alpha &= \frac{\hat{\gamma}}{9 - 4\sqrt{5}}, \\
\beta &= \frac{4\hat{\gamma}a}{5^{5/6}(9 - 4\sqrt{5})}, \\
\hat{\alpha} &= -\frac{a}{5^{5/12}}, \\
\hat{\beta} &= \frac{1 + \sqrt{5}}{4a}, \\
\rho_G &= a(\sqrt{5} - 1)^{1/3},
\end{aligned} \tag{12}$$

where  $a$  junction radius.

In this case the boundary surface entails via the field equations a jump in second derivations of metric coefficient, but first derivatives remains. Thus, from the junction conditions, the "interior" metric parameters can be determined at the boundary surface in terms of the "exterior" metric parameters. Note that for this case  $K_b^a$  is continuous across  $\Sigma$ . Hence, Darmois-Israel junction conditions are fulfilled. Such construction allows us to associate rigorously with both gauges as a same "space", which is a unique differentiable manifold  $M^4$ .

### 3 Discussion

The gravitational field equations define only the metric over manifold  $M^4$ , but not its topological property. The same system of reference in manifold  $M^4$  is interpreted as this or that co-ordinate system in pseudo - Riemannian space  $V^4(g)$  at a different selection of equations (2), giving some minimum of information about properties of the required metric. Nevertheless, Einstein's equations have, of course, a non-enumerable set of the solutions which are not possessing properties of the necessary metric. Generally it is impossible to be assured, that always it will be possible to coat  $V^4(g)$  with set of co-ordinate neighborhoods of the same class, i.e. featured by the same operators  $C(\mu)$ . It does necessary acceptance enough wide guesses of the nature of operators  $C(\mu)$ . The above consideration confirms the conclusion, that space  $V^4$  one can featured by two or more known class of co-ordinates. The solutions of Einstein's equations for such constructions one assume as "interior" and "exterior" one. Thus, the common geometry emerges from the junction conditions at the boundary surface. It must be combined with Darmois-Israel junction conditions. We show that our approach provides a clear way of showing that the Schwarzschild solution is not a unique static spherically symmetric solution, providing some incite on how the current form of Birkhoff's theorem breaks down. All results can be stated for four dimensional (pseudo) Riemannian

manifolds. It should also be noted that because general relativity is a highly non-linear theory, it is not always easy to understand what qualitative features solutions might possess, and here the composite class of solutions can be used as a guide.

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